A solvable model for non-additive stochastic processes

1984 J. Phys. A: Math. Gen. 17 175

(http://iopscience.iop.org/0305-4470/17/1/019)

J. Phys. A: Math. Gen. 17 (1984) 175–178. Printed in Great Britain

A solvable model for non-additive stochastic processes

W M Zheng†‡

Center for Studies in Statistical Mechanics, University of Texas at Austin, Austin, Texas 78712, USA

Received 25 May 1983, in final form 29 July 1983

Abstract. An exactly solvable model is proposed for multiplicative stochastic processes.

It has been pointed out that in externally driven systems the fluctuations of the 'pumping parameter' might play a very pronounced role when they depend on the specific state of the systems (Schenzle and Brand 1979, Horsthemke and Malek-Mansour 1976). Recently, in investigation of the pure death explosion process and the stochastic Schrögl model a so-called chemical explosion regime has been introduced (Baras *et al* 1982, Frankowitz and Nicolis 1983). The master equations studied for the chemical explosion regime can be approximated by differential equations, which are Fokker–Planck-type equations with non-constant diffusion coefficients. We think that the anomalous fluctuations due to the non-constant diffusion might be the origin of the regime.

A solvable model for a class of multiplicative stochastic processes has been found and solved in Schenzle and Brand (1979). The generalised Langevin equation considered there is

$$\dot{x} = \alpha x - x^{1+\gamma} + x\xi \tag{1}$$

where the Gaussian random force satisfies $\langle\xi\rangle = 0$ and $\langle\xi\xi_\tau\rangle = Q\delta(\tau)$ and α and γ are positive constants§. The time-dependent Fokker–Planck equation corresponding to the Langevin equation (1) is given by (Stratonovich 1963)

$$\dot{p} = (\partial/\partial x)\{[x^{1+\gamma} - (\alpha + \tfrac{1}{2}Q)x]p\} + \tfrac{1}{2}Q(\partial^2/\partial x^2)(x^2 p). \tag{2}$$

If we regard $\xi(t)$ as an ordinary function of time, equation (1) is the Bernoulli equation (Davis 1960), which can be reduced to a linear equation by means of the transformation $z = x^{-\gamma}$. Here we extend the model to consider the Langevin equation (in the Stratonovich sense)

$$\dot{x} = \alpha x + \beta x^{1+\gamma} + \xi(\alpha' x + \beta' x^{1+\gamma}). \tag{3}$$

After making the transformation $z = x^{-\gamma}$, we find from equation (3) the Langevin equation for z (Arnold 1973)

$$\dot{z} = -\gamma(\alpha z + \beta) - \gamma\xi(\alpha' z + \beta'). \tag{4}$$

† On leave from the Institute of Theoretical Physics, Academia Sinica, Beijing, China.

‡ Supported in part by the Robert A Welch Foundation.

§ This class of nonlinear stochastic models has been solved by a method of linear imbedding in Graham and Schenzle (1982); the long time decay constants have been discussed by Gardiner and Graham (1982).

Thus, the corresponding Fokker–Planck equation is

$$\partial\tilde{p}/\partial t=(\partial/\partial z)\{[\gamma(\alpha z+\beta)-\tfrac{1}{2}\alpha' Q\gamma^2(\alpha' z+\beta')]\tilde{p}\}+\tfrac{1}{2}Q\gamma^2(\partial^2/\partial z^2)[(\alpha' z+\beta')^2\tilde{p}]$$
$$\equiv(\partial/\partial z)[g(z)\tilde{p}]+\tfrac{1}{2}Q(\partial^2/\partial z^2)[h(z)\tilde{p}] \qquad (5)$$

where

$$\tilde{p}(z,t)=-p(x,t)(\mathrm{d}x/\mathrm{d}z)=\gamma^{-1}x^{1+\gamma}p(x,t). \qquad (6)$$

(In writing equation (6) we have assumed γ to be positive; for a negative γ, we can choose $\tilde{p}(z,t)=p(x,t)(\mathrm{d}x/\mathrm{d}z)$.)

It is easy to find the stationary solution to equation (5)

$$\tilde{p}_s=\frac{N'}{h(z)}\exp\left(-\frac{2}{Q}\int\frac{g(z)}{h(z)}\mathrm{d}z\right)$$
$$=N(\alpha' z+\beta')^{-2\alpha/Q\gamma\alpha'^2-1}\exp[-2(\alpha\beta'-\alpha'\beta)/Q\gamma\alpha'^2(\alpha' z+\beta')] \qquad (7)$$

where N' and N are the normalisation factors. To guarantee the existence of the stationary state, the parameters should satisfy some additional condition.

Introducing

$$q(z,t)=\tilde{p}(z,t)/\tilde{p}_s(z), \qquad (8)$$

from equation (5) we can derive the backward equation for $q(z,t)$

$$\partial q/\partial t=-g(z)\,\partial q/\partial z+\tfrac{1}{2}Qh(z)\,\partial^2 q/\partial z^2$$

or

$$\partial q/\partial t=[(\tfrac{1}{2}\alpha'^2Q\gamma^2-\gamma\alpha)Z+\gamma(\alpha\beta'-\alpha'\beta)]\,\partial q/\partial Z+\tfrac{1}{2}Q\gamma^2\alpha'^2Z^2\,\partial^2 q/\partial Z^2$$

where in the last step we have set $Z=\alpha' z+\beta'$.

By splitting off a time factor $\mathrm{e}^{-\lambda t}$, the eigenvalue equation can be obtained from equation (9)

$$\mathrm{d}^2\varphi/\mathrm{d}Z^2+(\Gamma/Z+\Delta/Z^2)\,\mathrm{d}\varphi/\mathrm{d}Z+(\Lambda/Z^2)\varphi=0 \qquad (10)$$

where

$$\Gamma=1-\frac{2\alpha}{Q\gamma\alpha'^2}, \qquad \Delta=\frac{2(\alpha\beta'-\alpha'\beta)}{Q\gamma\alpha'^2}, \qquad \Lambda=\frac{2\lambda}{Q\gamma^2\alpha'^2}.$$

Writing equation (10) in the form of the general confluent equation (Abramowitz and Stegun 1965)

$$\varphi''+\left(\frac{2A}{Z}+2f'+\frac{bh'}{h}-h'-\frac{h''}{h'}\right)\varphi'+\left[\left(\frac{bh'}{h}-h'+\frac{h''}{h'}\right)\left(\frac{A}{Z}+f'\right)\right.$$
$$\left.+\frac{A(A-1)}{Z^2}+\frac{2Af'}{Z}+f''+f'^2-\frac{ah'^2}{h}\right]\varphi=0 \qquad (11)$$

with $f=0$ and $h=\Delta/Z$, we obtain, for example, for positive α and γ the eigenfunctions and eigenvalues for the discrete spectrum

$$\Lambda_n=n(1-\Gamma-n), \qquad \text{for integral } n<\alpha/\gamma Q\alpha'^2, \qquad (12a)$$
$$\varphi_n=Z^{-n}\,{}_1F_1(-n,-2n+2-\Gamma,\Delta/Z) \qquad (12b)$$
$$=(-1)^n n!\,Z^{-n}L_n^{(-2n+1-\Gamma)}(\Delta/Z) \qquad (12c)$$

and for the continuous spectrum when $\Lambda > [\alpha/\gamma Q\alpha'^2]^2 \equiv \mu^2$

$$\varphi_\Lambda = Z^a U(a, b, \Delta/Z) \tag{13}$$

where

$$a = -\mu + \mathrm{i}(\Lambda - \mu^2)^{1/2}, \qquad b = 1 + 2\mathrm{i}(\Lambda - \mu^2)^{1/2}.$$

Therefore, expanding the solution in terms of the eigenfunctions, finally we find

$$p(x, t) = x^{-1-\gamma}\left\{Z^{\Gamma-2}\exp(-\Delta/Z)\sum \mathrm{e}^{-\lambda_n t} C_n \varphi_n(Z)\right\}_{Z=\alpha' z+\beta'} \tag{14}$$

where the C_n's are expansion coefficients determined from the initial condition. To find C_n, we need adjoint eigenfunctions, which can be obtained in a similar way. We will not discuss this here.

When expression (7) is not normalisable, the backward equation will encounter difficulties. However, the form of the solution (14) suggests that we choose $A = \Gamma - 2$, $f = \Delta/Z$ and $h = \Delta/Z$ to directly transform the eigenequation corresponding to equation (6)

$$Z^2\, \mathrm{d}^2\varphi/\mathrm{d}Z^2 + [(2-\Gamma)Z - \Delta]\, \mathrm{d}\varphi/\mathrm{d}Z + (\Lambda + 2 - \Gamma)\varphi = 0 \tag{15}$$

into the general confluent equation (11), and then find the solution, which is still of the form of (14). In this case, it might be troublesome to find expansion coefficients.

The model discussed includes the stochastic Verhulst equation (Goel *et al* 1971, Goel and Richter-Dyn 1974, Morita 1982) and the Suzuki–Kaneko–Sasagawa model (Suzuki *et al* (1980), but the solution of the FP equation is not given there) as particular cases.

To close the paper, we show another particular case, i.e. that discussed in Schenzle and Brand (1979): $\beta = -1$, $\alpha' = 1$ and $\beta' = 0$. In this case we have

$$Z = x^{-\gamma}, \qquad \Gamma = 1 - 2\alpha/Q\gamma, \qquad \Delta = 2/Q\gamma, \qquad \Lambda = 2\lambda/Q\gamma^2,$$

and hence, for example, from equation (14)

$$p_s(x) = x^{2\alpha/Q-1}\exp(-2x^\gamma/\gamma Q),$$

and from equation (12)

$$\lambda_n = \tfrac{1}{2}Q\gamma^2 n(2\alpha/Q\gamma - n) = n\gamma Q(\alpha/Q - \tfrac{1}{2}n\gamma),$$

$$\varphi_n = x^{-n\gamma}\,{}_1F_1(-n, -2n + 1 + 2/\gamma Q, 2x^\gamma/\gamma Q).$$

All the results will be found to coincide with those obtained for the case in Schenzle and Brand (1979).

References

Abramowitz M and Stegun I 1965 *Handbook of Mathematical Functions* (New York: Dover)
Arnold L 1973 *Stochastic Differential Equations* (New York: Wiley)
Baras F, Nicolis G, Malek-Mansour M and Turner J W 1982 *Stochastic Theory of Adiabatic Explosion, Preprint*
Davis H T 1960 *Introduction to Nonlinear Differential and Integral Equations* (Washington: USAEC)
Frankowitz M and Nicolis G 1983 *Transition evolution towards a unique stable state, Preprint*
Gardiner C W and Graham R 1982 *Phys. Rev.* A **25** 1851
Goel N S, Maitra S C and Montroll E W 1971 *Rev. Mod. Phys.* **43** 231

Goel N S and Richter-Dyn N 1974 *Stochastic Models in Biology* (New York: Academic)
Graham R and Schenzle A 1982 *Phys. Rev.* A **25** 1731
Horsthemke W and Malek-Mansour M 1976 *Z. Phys.* B **24** 307
Morita A 1982 *J. Chem. Phys.* **76** 4191
Schenzle A and Brand H 1979 *Phys. Lett.* **69A** 313
Stratonovich R L 1963 *Topics in the Theory of Random Noise* (New York: Gordon and Breach)
Suzuki M, Kaneko K and Sasagawa F 1980 *Prog. Theor. Phys.* **65** 828